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On the Nature of the Virasoro Algebra

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To the great algebraist Victor Kac, on occasion of his 55th birthday.

Abstract

The multiplication in the Virasoro algebra

$$[e_p, e_q] = (p-q)e_{p+q} + \theta\left(p^3 - p\right)\delta_{p+q}, \qquad p, q \in \mathbf{Z},$$
$$[\theta, e_p] = 0,$$

comes from the commutator $[e_p, e_q] = e_p * e_q - e_q * e_p$ in a quasiassociative algebra with the multiplication

$$e_p * e_q = -\frac{q(1+\epsilon q)}{1+\epsilon(p+q)} e_{p+q} + \frac{1}{2}\theta\left[p^3 - p + \left(\epsilon - \epsilon^{-1}\right)p^2\right]\delta^0_{p+q}, \qquad (*)$$
$$e_p * \theta = \theta * e_p = 0.$$

The multiplication in a quasiassociative algebra $\mathcal{R}$ satisfies the property

$$a*(b*c) - (a*b)*c = b*(a*c) - (b*a)*c, \qquad a, b, c \in \mathcal{R}. \qquad (**)$$

This property is necessary and sufficient for the Lie algebra $Lie(\mathcal{R})$ to have a phase space. The above formulae are put into a cohomological framework, with the relevant complex being different from the Hochschild one even when the relevant quasiassociative algebra $\mathcal{R}$ becomes associative. Formula $(*)$ above also has a differential-variational counterpart.

1 Introduction

Quasiassociative algebras, originally discovered by Vinberg [8]–[10] and Koszul [3] in the 1960's in the study of homogeneous convex cones, appear also as an underlying structure of those Lie algerbras that possess a phase space. Namely, for a given Lie algebra $\mathcal{G}$, the following three conditions are equivalent [5]:

(i) $\mathcal{G} = Lie(\mathcal{R})$ for some quasiassociative algebra $\mathcal{R}$;

(ii) Let $\rho : \mathcal{G} \to \mathrm{End}(\mathcal{G}^*)$ be a representation, not necessarily coadjoint one, such that on the semidirect sum Lie algebra $\mathcal{G} \underset{\rho}{\ltimes} \mathcal{G}^*$, the symplectic form is a 2-cocycle;

(iii) The natural Poisson bracket on the Lie algebra $\mathcal{G} \underset{\rho}{\ltimes} \mathcal{G}^*$ is compatible with the canonical Poisson bracket.

Thus, the quasiassociative algebras form a natural category from the point of view of Classical and Quantum mechanics. A list of Lie algebras with a phase space, given in [5], includes such non-evident cases as Lie algebras of vector fields on $\mathbf{R}^n$ and current algebras. One of the principal Lie algebras of physical interest, the Virasoro algebra, has been, however, conspiciously under-privileged so far. Its underlying quasiassociative structure is treated in the next two Sections. (Still more Lie algebras with a phase space can be found in Chapter 2 in [7].)

Before leaving the phase-space perspective for more mathematical matters, let me make two comments. First, the category of quasiassociative algebras is closed with respect to the operation of phase-space extension, unlike the smaller category of associative algebras: if $\mathcal{R}$ is quasiassociative then so is $\mathrm{T}^*\mathcal{R}$, where multiplication in $\mathrm{T}^*\mathcal{R}$ is given by the formula [5]

$$\begin{pmatrix} x \\ \bar{x} \end{pmatrix} * \begin{pmatrix} y \\ \bar{y} \end{pmatrix} = \begin{pmatrix} x * y \\ x * \bar{y} \end{pmatrix}, \qquad x, y \in \mathcal{R}, \quad \bar{x}, \bar{y} \in \mathcal{R}^* = \mathrm{Hom}(\mathcal{R}, \dots), \tag{1.1a}$$

$$\langle x * \bar{y}, y \rangle = -\langle \bar{y}, x * y \rangle. \tag{1.1b}$$

Second, if $\rho : \mathcal{G} \to \mathrm{End}(\mathcal{G}^*)$ is the representation staring in the properties (ii) and (iii) above, then the associated quasiassociative multiplication on $\mathcal{G}$ is given by the formula

$$x * y = \rho^d(x)(y), \tag{1.2}$$

where $\rho^d : \mathcal{G} \to \mathrm{Eng}(\mathcal{G})$ is the representation dual to ρ. The condition for the symplectic form on $\mathcal{G} \underset{\rho}{\ltimes} \mathcal{G}^*$ to be a 2-cocycle is then equivalent to the property

$$\rho^d(x)(y) - \rho^d(y)(x) = [x, y], \qquad \forall\, x, y \in \mathcal{G}. \tag{1.3}$$

Thus,

$$Lie(\mathrm{T}^*\mathcal{R}) = \mathrm{T}^* Lie(\mathcal{R}). \tag{1.4}$$

The equation (1.3) appears also in a very different context, as the condition for the complex of differential forms on the Universal Enveloping Algebra $U(\mathcal{G})$ to be ghost-free (see [6], equations (7.4) and (7.5).)

Turning back to the Virasoro algebra, we see from formula $(*)$ that we have what appears to be a central extension of the corresponding centerless quasiassociative multiplication

$$e_p * e_q = -\frac{q(1+\epsilon q)}{1+\epsilon(p+q)} e_{p+q}, \qquad p, q \in \mathbf{Z}, \tag{1.5}$$

where ϵ can be treated as either a formal parameter or a number such that $\epsilon^{-1} \bar{\in} \mathbf{Z}$. The next Section contains a quick verification that formula (1.5) satisfies the quasiassociativity

property (∗∗). Section 3 is devoted to central extensions of quasiassociative algebras in general and the algebra (1.5) in particular, resulting in the formula (∗) from the Abstract. In Section 4 we re-interpret in the language of 2-cocycles the property of a bilinear form to provide a central extension of a quasiassociative algebra; this interpretation then leads to a complex on the space of cochains $C^n = \mathrm{Hom}(\mathcal{R}^{\otimes n}, \cdot)$. Section 5 generalizes this complex to the case $C^n = \mathrm{Hom}(\mathcal{R}^{\otimes n}, \mathcal{M})$, where $\mathcal{R}$ acts nontrivally on $\mathcal{M}$. Section 6 deals with the dual objects, homology. The last Section 7 is devoted to differential-variational versions of the preceding results, for the case when the centerless Virasoro algebra is replaced by the Lie algebra of vector fields on the circle with the commutator

$$[X, Y] = XY' - X'Y, \qquad ' = d/dz, \tag{1.6}$$

and the central extension is given by the Gelfand-Fuks 2-cocycle

$$\omega(X, Y) = \int XY''' \, dz. \tag{1.7}$$

Appendix 1 contains a short proof that the Virasoro algebra does not come from an associative one. Semi-direct sums of quasiassociative algebras are treated in Appendix 2. In Appendix 3 we prove that if G is a connected Lie group whose Lie algebra $\mathcal{G}$ comes out of a quasiassociative algebra then the Lie algebra $\mathcal{D}(G)$ of vector fields on G also allows a quasiassociative representation.

2 The Centerless Virasoro Algebra

Suppose a space with a basis $\{e_p | p \in G$, a commutative ring$\}$ has the multiplication of the form

$$e_p * e_q = f(p, q) e_{p+q}. \tag{2.1}$$

Then

$$\begin{aligned} e_p * (e_q * e_r) - (e_p * e_q) * e_r &= f(q, r) e_p * e_{q+r} - f(p, q) e_{p+q} * e_r \\ &= [f(q, r) f(p, q + r) - f(p, q) f(p + q, r)] e_{p+q+r}, \end{aligned} \tag{2.2}$$

so that the quasiassociativity condition (∗∗), the symmetry between p and q, is equivalent to the relation

$$f(q, r) f(p, q + r) - f(p, q) f(p + q, r) = f(p, r) f(q, p + r) - f(q, p) f(p + q, r), \tag{2.3}$$

which can be rewritten as

$$[f(p, q) - f(q, p)] f(p + q, r) = f(q, r) f(p, q + r) - f(p, r) f(q, p + r). \tag{2.4}$$

By formula (1.5),

$$f(p, q) = -\frac{q(1 + \epsilon q)}{1 + \epsilon(p + q)}, \tag{2.5}$$

and we have to check that this $f(p, q)$ satisfies formula (2.4).

First,

$$f(p,q) - f(q,p) = \frac{1}{1+\epsilon(p+q)}[-q(1+\epsilon q) + p(1+\epsilon p)] = p - q, \tag{2.6}$$

so that

$$e_p * e_q - e_q * e_p = (p-q)e_{p+q}, \tag{2.7}$$

guaranteeing that the Lie algebra generated by formula (1.5) is indeed the centerless Virasoro algebra.

Now, for the LHS of formula (2.4) we obtain

$$(p-q)\frac{-r(1+\epsilon r)}{1+\epsilon(p+q+r)}, \tag{2.8ℓ}$$

while for the RHS of formula (2.4) we get

$$\begin{aligned} &\frac{-r(1+\epsilon r)}{1+\epsilon(q+r)} \cdot \frac{-(q+r)[1+\epsilon(q+r)]}{1+\epsilon(p+q+r)} - \frac{-r(1+\epsilon r)}{1+\epsilon(p+r)} \cdot \frac{-(p+r)[1+\epsilon(p+r)]}{1+\epsilon(p+q+r)} \\ &\quad = \frac{-r(1+\epsilon r)}{1+\epsilon(p+q+r)}[-(q+r)+(p+r)] = \frac{-r(1+\epsilon r)}{1+\epsilon(p+q+r)}(p-q), \end{aligned} \tag{2.8r}$$

and this is the same as formula (2.8ℓ).

Remark 2.9. Formula (2.5) is not the only solution of the equation (2.4) satisfying the Lie boundary condition

$$f(p,q) - f(q,p) = p - q. \tag{2.10}$$

For example,

$$f(p,q) = \lambda - q, \qquad \lambda = \text{const}, \tag{2.11}$$

is also a solution. It does not alow a proper central extension, however.

3 Central Extensions of Quasiassociative Algebras

Let K be a commutative ring over which our quasiassociative algebra $\mathcal{R}$ is an algebra. Let $\Omega : \mathcal{R} \times \mathcal{R} \to K$ be a bilinear form. It defines a multiplication on the space $\tilde{\mathcal{R}} = \mathcal{R} \oplus K$, by the rule

$$\begin{pmatrix} a \\ \alpha \end{pmatrix} * \begin{pmatrix} b \\ \beta \end{pmatrix} = \begin{pmatrix} a * b \\ \Omega(a,b) \end{pmatrix}, \qquad a, b \in \mathcal{R}, \quad \alpha, \beta \in K. \tag{3.1}$$

When is $\tilde{\mathcal{R}}$ quasiassociative? We have:

$$\begin{aligned} &\begin{pmatrix} a \\ \alpha \end{pmatrix} * \left(\begin{pmatrix} b \\ \beta \end{pmatrix} * \begin{pmatrix} c \\ \gamma \end{pmatrix} \right) - \left(\begin{pmatrix} a \\ \alpha \end{pmatrix} * \begin{pmatrix} b \\ \beta \end{pmatrix} \right) * \begin{pmatrix} c \\ \gamma \end{pmatrix} \\ &= \begin{pmatrix} a \\ \alpha \end{pmatrix} * \begin{pmatrix} b * c \\ \Omega(b,c) \end{pmatrix} - \begin{pmatrix} a * b \\ \Omega(a,b) \end{pmatrix} * \begin{pmatrix} c \\ \gamma \end{pmatrix} = \begin{pmatrix} a * (b * c) - (a * b) * c \\ \Omega(a, b * c) - \Omega(a * b, c) \end{pmatrix}. \end{aligned} \tag{3.2}$$

Thus, $\tilde{\mathcal{R}}$ is quasiassociative iff

$$\Omega(a, b * c) - \Omega(a * b, c) = \Omega(b, a * c) - \Omega(b * a, c). \tag{3.3}$$

This can be equivalently rewritten as

$$\Omega(b, a * c) - \Omega(a, b * c) + \Omega([a, b], c) = 0, \tag{3.4}$$

where $[a, b] = a * b - b * a$ is the commutator in the Lie algebra $Lie(\mathcal{R})$. By construction, the bilinear form

$$\omega(a, b) = \Omega(a, b) - \Omega(b, a) \tag{3.5}$$

defines a central extension of the Lie algebra $Lie(\mathcal{R})$; thus, ω is a 2-cocycle on this Lie algebra.

While we are at it, let's look at *trivial* central extensions of $\mathcal{R}$. These are produced from the multiplication

$$\begin{pmatrix} a \\ \alpha \end{pmatrix} \underset{t}{*} \begin{pmatrix} b \\ \beta \end{pmatrix} = \begin{pmatrix} a * b \\ 0 \end{pmatrix} \tag{3.6}$$

by linear transformations of the form

$$\Phi = \begin{pmatrix} id & 0 \\ \langle u, \cdot \rangle & 1 \end{pmatrix}, \qquad u \in \mathcal{R}^*. \tag{3.7}$$

Thus, trivial extensions look like

$$\begin{aligned} \begin{pmatrix} a \\ \alpha \end{pmatrix} * \begin{pmatrix} b \\ \beta \end{pmatrix} &= \Phi \left(\Phi^{-1} \begin{pmatrix} a \\ \alpha \end{pmatrix} \underset{t}{*} \Phi^{-1} \begin{pmatrix} b \\ \beta \end{pmatrix} \right) \\ &= \Phi \left(\begin{pmatrix} a \\ \cdots \end{pmatrix} \underset{t}{*} \begin{pmatrix} b \\ \cdots \end{pmatrix} \right) = \Phi \begin{pmatrix} a * b \\ 0 \end{pmatrix} = \begin{pmatrix} a * b \\ \langle u, a * b \rangle \end{pmatrix}, \end{aligned} \tag{3.8}$$

so that trivial "2-cocycles" on $\mathcal{R}$ are of the form

$$\Omega(a, b) = \langle u, a * b \rangle, \qquad u \in \mathcal{R}^*. \tag{3.9}$$

The award of the title "cocycle" to Ω will be justified in the next Section, where the criterion (3.4) is recast as

$$\delta\Omega(a, b, c) = 0. \tag{3.10}$$

Similarly, central extensions differing by a trivial 2-cocycle are equivalent: if

$$\begin{pmatrix} a \\ \alpha \end{pmatrix} \underset{1}{*} \begin{pmatrix} b \\ \beta \end{pmatrix} = \begin{pmatrix} a * b \\ \omega(a, b) + \langle u, a * b \rangle \end{pmatrix}$$

and

$$\begin{pmatrix} a \\ \alpha \end{pmatrix} \underset{2}{*} \begin{pmatrix} b \\ \beta \end{pmatrix} = \begin{pmatrix} a * b \\ \omega(a, b) \end{pmatrix}$$

are two such extensions, then the transformation Φ (3.7) takes the second multiplication into the first one.

Let us return to the case of the Virasoro algebra. Formula $(*)$ shows that we have a central extension

$$\Omega(e_p, e_q) = \varphi(p)\delta^0_{p+q}. \tag{3.11}$$

$$2\varphi(p) = p^3 - \epsilon^{-1}p^2 - p + \epsilon p^2 = -p(1-\epsilon p)\left(1+\epsilon^{-1}p\right). \tag{3.12}$$

The condition (3.4), in the notation (2.1) and (3.11), becomes:

$$\begin{aligned} 0 &= \Omega(e_q, e_p * e_r) - \Omega(e_p, e_q * e_r) + \Omega([e_p, e_q], e_r) \\ &= f(p,r)\Omega(e_q, e_{p+r}) - f(q,r)\Omega(e_p, e_{q+r}) + (p-q)\Omega(e_{p+q}, e_r) \\ &= [f(p,r)\varphi(q) - f(q,r)\varphi(p) + (p-q)\varphi(p+q)]\delta^0_{p+q+r}, \end{aligned} \tag{3.13}$$

which can be rewritten as

$$(p-q)\varphi(p+q) = \varphi(p)f(q,-p-q) - \varphi(q)f(p,-p-q). \tag{3.14}$$

With $f(p,q)$ and $\varphi(p)$ given by formula (2.5) and (3.12) respectively, for the $2 \times LHS$ of formula (3.14) we get:

$$-(p-q)(p+q)[1-\epsilon(p+q)]\left[1+\epsilon^{-1}(p+q)\right], \tag{3.15ℓ}$$

while for the $2 \times RHS$ of formula (3.14) we obtain:

$$\begin{aligned} &-p(1-\epsilon p)\left(1+\epsilon^{-1}p\right)\frac{(p+q)[1-\epsilon(p+q)]}{1-\epsilon p} \\ &\quad +q(1-\epsilon q)\left(1+\epsilon^{-1}q\right)\frac{(p+q)[1-\epsilon(p+q)]}{1-\epsilon q} \\ &\quad = (p+q)[1-\epsilon(p+q)](q-p)\left[1+\epsilon^{-1}(p+q)\right], \end{aligned} \tag{3.15r}$$

and this is the same as the expression (3.15ℓ).

Thus, we get a central extension of the quasiassociative algebra (2.1), (2.5). It remains to check that the 2-cocycle ω (3.5) is indeed the one entering the Virasoro algebra. We have:

$$\begin{aligned} \omega(e_p, e_q) &= \Omega(e_p, e_q) - \Omega(e_q, e_p) = [\varphi(p) - \varphi(q)]\delta^0_{p+q} \\ &= \frac{1}{2}\left\{\left[p^3 - p + \left(\epsilon - \epsilon^{-1}\right)p^2\right] - \left[(-p)^3 - (-p) + \left(\epsilon - \epsilon^{-1}\right)(-p)^2\right]\right\}\delta^0_{p+q} \\ &= \left(p^3 - p\right)\delta^0_{p+q}. \end{aligned} \tag{3.16}$$

4 The Quasiassociative Complex

Let $\mathcal{M}$ be a K-module. Define the cochains on $\mathcal{R}$ with values of $\mathcal{M}$ as

$$C^n = C^n(\mathcal{R}, \mathcal{M}) = \mathrm{Hom}_K(\mathcal{R}^{\otimes n}, \mathcal{M}), \qquad n \in \mathbf{N}; \quad C^0 := \mathcal{M}. \tag{4.1}$$

In the preceding Section we in effect met two coboundary operators $\delta : C^n \to C^{n+1}$ for $n = 1$ and $n = 2$, in formulae (3.9) and (3.4) respectively:

$$\psi \in C^1 \quad \Rightarrow \quad \delta\psi(a_1, a_2) = \psi(a_1 * a_2), \tag{4.2}$$

$$\psi \in C^2 \quad \Rightarrow \quad \delta\psi(a_1, a_2, a_3) = \psi(a_2, a_1 * a_3) - \psi(a_1, a_2 * a_3) + \psi([a_1, a_2], a_3). \quad (4.3)$$

It is obvious that $\delta^2 = 0$ on C^1, and the roundabout way this equality was verified in the preceeding Section actually proves that

$$H^2(\mathcal{R}) := H^2(\mathcal{R}, K) \quad (4.4)$$

describes the K-module of isomorphism classes of 1-dimensional central extensions of $\mathcal{R}$ by K.

Guided by formulas (4.2) and (4.3), we define the coboundary operator $\delta : C^n \to C^{n+1}$ for all $n \in \mathbf{Z}_+$, as follows

$$\delta = 0 \quad \text{on} \quad C^0; \quad (4.5)$$

$$\psi \in C^n, \quad n \geq 1 \quad \Rightarrow$$

$$\delta\psi(a_1, \ldots, a_n, a) = \sum_{i=1}^{n} (-1)^{i+1} \psi(\ldots \hat{i} \ldots, a_i * a) + \quad (4.6a)$$

$$+ \sum_{1 \leq i < j \leq n} (-1)^{i+j+1} \psi([a_i, a_j] \ldots \hat{i} \ldots \hat{j} \ldots a). \quad (4.6b)$$

The hat over the argument signifies this argument's absence; the last sum (4.6b) is missing when $n < 2$; the right-most argument, $a = a_{n+1}$, is considered on a different footing from the rest, $a_1, \ldots, a_n$. (We see that $H^1(\mathcal{R}) = \{\mathrm{Ann}(\mathcal{R} * \mathcal{R}) \subset \mathcal{R}^*\}$.)

Before proceeding further, we need to make some minimal skewsymmetry observations.

Definition 4.7. *Suppose $n \geq 3$. If κ is such that $2 \leq \kappa < n$, then a cochain $\psi \in C^n$ is called κ-skewsymmetric if it is skewsymmetric in its first κ arguments.*

Proposition 4.8. *(i) For $n = 2$, $\delta(C^n)$ is 2-skewsymmetric;*
(ii) Suppose $n \geq 3$ and $\psi \in C^n$; if ψ is κ-skewsymmetric then so is $\delta\psi$.

Proof. (i) Formula (4.3) makes the claim obvious for $n = 2$;
(ii) For $n \geq 3$, the sums (4.6a) and (4.6b) each change sign under the transposition $(i, i+1)$ for all $i < \kappa$. ∎

Thereafter we assume that all our cochains are κ-skewsymmetric for some fixed $\kappa \geq 2$.

Proposition 4.9. $\delta^2 = 0$ *on 2-skewsymmetric cochains.*

Proof. Let $\psi \in C^n, n \geq 2$. Set $\nu = \delta\psi$:

$$\nu(a_1, \ldots, a_n, z) = \sum_{i=1}^{n} (-1)^{i+1} \psi(\ldots \hat{i} \ldots, iz) + \sum_{1 \leq i < j \leq n} (-1)^{i+j+1} \psi([i, j] \ldots \hat{i} \ldots \hat{j} \ldots, z),$$

where for brevity we write iz insted of $a_i * z$, and $[i, j]$ instead of $[a_i, a_j]$. Then,

$$\delta\nu(y_1, \ldots, y_{n+1}, t)$$

$$= \sum_{s=1}^{n+1} (-1)^{s+1} \nu(\hat{s}, st) + \quad (4.11a)$$

$$+\sum_{1\le p<q\le n+1}(-1)^{p+q+1}\nu[(p,q]\hat{p}\hat{q},t), \tag{4.11b}$$

where for further brefity we now suppress the "..." convention.

We shall work out separately the expression (4.11a) and (4.11b).

(a) We have:

$$\begin{aligned}\nu(\hat{s},t) &= \sum_{i<s}(-1)^{i+1}\psi(\hat{i}\hat{s},i(st)) + \sum_{i>s}(-1)^{i}\psi(\hat{s}\hat{i},i(st))\\ &+ \sum_{i<j<s}(-1)^{i+j+1}\psi([i,j]\hat{i}\hat{j}\hat{s},st)\\ &+ \sum_{i<s,j>s}(-1)^{i+j}\psi([i,j]\hat{i}\hat{s}\hat{j},st) + \sum_{s<i<j}(-1)^{i+j+1}\psi([i,j]\hat{s}\hat{i}\hat{j},st).\end{aligned} \tag{4.12}$$

Multiplying all this by $(-1)^{s+1}$ and summing on s, we get

$$\{(4.11a)\} =$$

$$= \sum_{a<b}(-1)^{a+b}\{\psi(\hat{a}\hat{b},a(bt)) \tag{4.13a}$$

$$-\psi(\hat{a}\hat{b},b(at))\} \tag{4.13b}$$

$$+\sum_{a<b<c}(-1)^{a+b+c}\{\psi([a,b]\hat{a}\hat{b}\hat{c},ct) \tag{4.13c}$$

$$-\psi([a,c]\hat{a}\hat{b}\hat{c},bt) \tag{4.13d}$$

$$+\psi([b,c]\hat{a}\hat{b}\hat{c},at)\}; \tag{4.13e}$$

(b) We have:

$$\begin{aligned}\nu([p,q]\hat{p}\hat{q},t) &= \psi(\hat{p}\hat{q},[p,q]t) + \sum_{\alpha<p}(-1)^{\alpha}\psi([p,q])\hat{\alpha}\hat{p}\hat{q},\alpha t)\\ &+ \sum_{p<\alpha<q}(-1)^{\alpha+1}\psi([p,q]\hat{p}\hat{\alpha}\hat{q},\alpha t) + \sum_{\alpha>q}(-1)^{\alpha}\psi([p,q]\hat{p}\hat{q}\hat{\alpha},\alpha t)\\ &+ \sum_{\alpha<p}(-1)^{\alpha+1}\psi([[p,q],\alpha]\hat{\alpha}\hat{p}\hat{q},t) + \sum_{p<\alpha<q}(-1)^{\alpha}\psi([[p,q],\alpha]\hat{p}\hat{\alpha}\hat{q},t)\\ &+ \sum_{\alpha>q}(-1)^{\alpha+1}\psi([[p,q],\alpha]\hat{p}\hat{q}\hat{\alpha},t) + \sum_{i<j<p}(-1)^{i+j+1}\psi([i,j],[p,q]\hat{i}\hat{j}\hat{p}\hat{q},t)\\ &+ \sum_{i<p,p<j<q}(-1)^{i+j}\psi[i,j],[p,q]\hat{i}\hat{p}\hat{j}\hat{q},t) + \sum_{i<p,j>q}(-1)^{i+j+1}\psi([i,j],[p,q]\hat{i}\hat{p}\hat{q}\hat{j},t)\\ &+ \sum_{p<i<j<q}(-1)^{i+j+1}\psi([i,j],[p,q]\hat{p}\hat{i}\hat{j}\hat{q},t) + \sum_{p<i<q<j}(-1)^{i+j}\psi([i,j],[p,q]\hat{p}\hat{i}\hat{q}\hat{j},t)\\ &+ \sum_{q<i<j}(-1)^{i+j+1}\psi([i,j],[p,q]\hat{p}\hat{q}\hat{i}\hat{j},t).\end{aligned}$$

Multiplying this monstrocity by $(-1)^{p+q+1}$ and summing on $\{1 \le p < q \le n+1\}$, we find:

$$\{(4.11b)\} =$$

$$= -\sum_{a<b} (-1)^{a+b} \psi(\hat{a}\hat{b}, [a,b]t) \tag{4.14a}$$

$$+ \sum_{a<b<c} (-1)^{a+b+c} \{-\psi([b,c]\hat{a}\hat{b}\hat{c}, at) \tag{4.14b}$$

$$+ \psi([a,c]\hat{a}\hat{b}\hat{c}, bt) \tag{4.14c}$$

$$- \psi([a,b]\hat{a}\hat{b}\hat{c}, ct) \tag{4.14d}$$

$$+ \psi([[b,c],a]\hat{a}\hat{b}\hat{c}, t) \tag{4.14e}$$

$$- \psi([[a,c],b]\hat{a}\hat{b}\hat{c}, t) \tag{4.14f}$$

$$+ \psi([[a,b],c]\hat{a}\hat{b}\hat{c}, t)\} \tag{4.14g}$$

$$+ \sum_{a<b<c<d} (-1)^{a+b+c+d} \{\psi([a,b],[c,d]\hat{a}\hat{b}\hat{c}\hat{d}, t) \tag{4.14h}$$

$$- \psi([a,c],[b,d]\hat{a}\hat{b}\hat{c}\hat{d}, t) \tag{4.14i}$$

$$+ \psi([a,d],[b,c]\hat{a}\hat{b}\hat{c}\hat{d}, t) \tag{4.14j}$$

$$+ \psi([b,c],[a,d]\hat{a}\hat{b}\hat{c}\hat{d}, t) \tag{4.14k}$$

$$- \psi([b,d],[a,c]\hat{a}\hat{b}\hat{c}\hat{d}, t) \tag{4.14l}$$

$$+ \psi([o,d][a,b]\hat{a}\hat{b}\hat{c}\hat{d}, t)\}. \tag{4.14m}$$

Grouping various terms together, we organize the cancellation scheme as follows:

1) (4.13a,b) and (4.14a), because of the equality

$$a(bt) - b(at) = [a,b]t \tag{4.15}$$

being the defining relation $(**)$ of a quasiassociative algebra;

2) (4.13c) and (4.14d); (4.13d) and (4.14c); (4.13e) and (4.14b);

3) (4.14e,f,g) by virtue of the Jacobi identity;

4) (4.14h,m); (4.14i,l); (4.14j,k); – all by virtue of ψ being 2-skewsymmetric. ■

5 The Quasiassociative Complex with Values in a Module

In this Section we generalize the coboundary operator $\delta : C^n \to C^{n+1}$ given by formula (4.6), to the case where the quasiassociative algebra $\mathcal{R}$ acts nontrivially on $\mathcal{M}$, the space where cochains take values.

Suppose $\chi : \mathcal{R} \to \mathrm{End}(\mathcal{M})$ is a linear map. It is natural to call it a representation of $\mathcal{R}$ if it behaves the way the left multiplication in $\mathcal{R}$ does:

$$\chi(a)\chi(b) - \chi(a * b) = \chi(b)\chi(a) - \chi(b * a), \qquad \forall\, a, b \in \mathcal{R}. \tag{5.1}$$

Since this can be rewritten as

$$[\chi(a), \chi(b)] = \chi([a, b]), \tag{5.2}$$

we simply have a representation of the underlying Lie algebra $Lie(\mathcal{R})$. It is interesting that for the purpose of extending the chain complex (4.6) of the preceeding Section, this natural and proper definition is insufficient; a stronger one is required. This insufficiency can be seen as follows.

Let $\psi : \mathcal{R} \to \mathcal{M}$ be a 1-cochain. By formula (4.2), we should now have

$$\delta\psi(a, b) = \psi(a * b) + c_1\chi(a)\psi(b) + c_2\chi(b)\psi(a), \tag{5.3}$$

with some constants c_1 and c_2. If we fix $m \in C^0 = \mathcal{M}$ and consider the natural definition for the operator $\delta : C^0 \to C^1$,

$$\delta m(a) = \chi(a)(m), \tag{5.4}$$

then

$$\begin{aligned} \left(\delta^2 m\right)(a, b) &= \delta m(a * b) + c_1\chi(a)\delta m(b) + c_2\chi(b)\delta m(a) \\ &= (\chi(a * b) + c_1\chi(a)\chi(b) + c_2\chi(b)\chi(a))(m). \end{aligned} \tag{5.5}$$

This expression has no reasons to vanish unless we change the definition of (left) representation to read

$$\chi(a * b) = \chi(a)\chi(b), \qquad \forall\, a, b \in \mathcal{R}, \tag{5.6}$$

and set $c_1 = -1$, $c_2 = 0$ in formula (5.3). (We can also adapt the dual point of view, defining (right) representation by the condition

$$\chi(a * b) = -\chi(b)\chi(a), \tag{5.7}$$

and setting $c_1 = 0$, $c_2 = 1$ in formula (5.3). But we won't pursue this avenue here, leaving it to the next Section.) Thus,

$$\delta\psi(a, b) = \psi(a * b) - a.\psi(b), \tag{5.8}$$

where

$$a.(\cdot) := \chi(a)(\cdot). \tag{5.9}$$

All told, we define the coboundary operator $\delta :\ C^n \to C^{n+1}$ by the formula

$$\begin{aligned}\delta\psi(a_1,\dots,a_n,a) &= \sum_{i=1}^{n}(-1)^{i+1}[\psi(\dots\hat{i}\dots,a_i * a) - a_i.\psi(\dots\hat{i}\dots,a)]\\ &\quad + \sum_{1\le i<j\le n}(-1)^{i+j+1}\psi([a_i,a_j]\dots\hat{i}\dots\hat{j}\dots,a).\end{aligned} \tag{5.10}$$

For $n=0$, formula (5.10) is to be understood as

$$\delta\psi(a_1) = -a_1.\psi, \qquad \psi\in\mathcal{M}. \tag{5.11}$$

The new extra sum in formula (5.10) doesn't destroy the property of δ to preserve κ-skewsymmeetry.

Proposition 5.12. $\delta^2 = 0$.

Proof. We have seen above that $\delta^2=0$ on C^0, and it's easy to verify that $\delta^2=0$ on C^1 and C^2. So let $n\ge 3$.

Setting

$$\delta = \delta^{\mathrm{old}} + \delta^{\mathrm{new}}, \tag{5.13}$$

where δ^{old} is given by formula (4.6), and δ^{new} is given by formula

$$\delta^{\mathrm{new}}\psi(a_1,\dots,a_n,a) = \sum_{i=1}^{n}(-1)^i a_i.\psi(\dots\hat{i}\dots,a), \tag{5.14}$$

we have for $\nu=\delta\psi$:

$$\begin{aligned}\nu(a_1,\dots,a_n,z) &= \sum_{i=1}^{n}(-1)^{i+1}\psi(\hat{i},iz)\\ &\quad + \sum_{i<j}(-1)^{i+j+1}\psi([i,j]\hat{i}\hat{j},z) + \sum_{i=1}^{n}(-1)^i a_i.\psi(\hat{i},z),\end{aligned} \tag{5.15}$$

$$\begin{aligned}\delta\nu(y_1,\dots,y_{n+1},t) &= \sum_{s=1}^{n+1}(-1)^{s+1}\nu(\hat{s},st)\\ &\quad + \sum_{p<q}(-1)^{p+q+1}\nu([p,q]\hat{p}\hat{q},t) + \sum_{\ell=1}^{n+1}(-1)^{\ell}y_\ell.\nu(\hat{\ell},t).\end{aligned} \tag{5.16}$$

We shall work out separately each of the three sums in the expression (5.16); since we have already verified in the preceding Section that $(\delta^{\mathrm{old}})^2=0$, we shall only keep track of the extra terms coming out of the operator $\delta^{\mathrm{old}}\delta^{\mathrm{new}} + \delta^{\mathrm{new}}\delta^{\mathrm{old}} + (\delta^{\mathrm{new}})^2$.

(a) We have:

$$\nu(\hat{s},st) \doteq \sum_{i<s}(-1)^i y_i.\psi(\hat{i}\hat{s},st) + \sum_{i>s}(-1)^{i+1}y_i.\psi(\hat{s}\hat{i},st).$$

Multiplying this by $(-1)^{s+1}$ and summing on s, we get

$$\{(5.16\text{a})\} \doteq \sum_{a<b}(-1)^{a+b}\left[-y_a.\psi(\hat{a}\hat{b}, bt) + y_b.\psi(\hat{a}\hat{b}, at)\right]; \tag{5.17}$$

(b) We have:

$$\nu([p,q]\hat{p}\hat{q}, t) \doteq -[y_p, y_q].\psi(\hat{p}\hat{q}, t) + \sum_{\alpha<p}(-1)^{\alpha+1}y_\alpha.\psi([p,q]\hat{\alpha}\hat{p}\hat{q}, t)$$

$$+ \sum_{p<\alpha<q}(-1)^{\alpha}y_\alpha.\psi([p,q]\hat{p}\hat{\alpha}\hat{q}, t) + \sum_{\alpha>q} y_\alpha.\psi([p,q]\hat{p}\hat{q}\hat{\alpha}, t).$$

Multiplying this by $(-1)^{p+q+1}$ and summing on $\{p<q\}$, we find:

$$\{(5.16\text{b})\} \doteq \sum_{p<q}(-1)^{p+q}[y_p, y_q].\psi(\hat{p}\hat{q}, t) \tag{5.18}$$

$$+ \sum_{a<b<c}(-1)^{a+b+c}\{y_a.\psi([b,c]\hat{a}\hat{b}\hat{c}, t) \tag{5.19a}$$

$$- y_b.\psi([a,c]\hat{a}\hat{b}\hat{c}, t) \tag{5.19b}$$

$$+ y_c.\psi([a,b]\hat{a}\hat{b}\hat{c}, t)\}; \tag{5.19c}$$

(c) We have:

$$y_\ell.\nu(\hat{\ell}, t) = y_\ell.\left\{\sum_{i<\ell}(-1)^{i+1}\psi(\hat{i}\hat{\ell}, it) + \sum_{i>\ell}(-1)^{i}\psi(\hat{\ell}\hat{i}, it)\right.$$

$$+ \sum_{i<j<\ell}(-1)^{i+j+1}\psi([i,j]\hat{i}\hat{j}\hat{\ell}, t) + \sum_{i<\ell<j}(-1)^{i+j}\psi([i,j]\hat{i}\hat{\ell}\hat{j}, t)$$

$$\left.+ \sum_{\ell<i<j}(-1)^{i+j+1}\psi([i,j]\hat{\ell}\hat{i}\hat{j}, t) + \sum_{i<\ell}(-1)^{i}y_i.\psi(\hat{i}\hat{\ell}, t) + \sum_{i>\ell}(-1)^{i+1}y_i.\psi(\hat{\ell}\hat{i}, t)\right\}.$$

Multiplying all this by $(-1)^{\ell}$ and summing on ℓ, we obtain:

$$\{(5.16\text{c})\} = \sum_{a<b}(-1)^{a+b}\{-y_b.\psi(\hat{a}\hat{b}, at) + y_a.\psi(\hat{a}\hat{b}, bt)\} \tag{5.20}$$

$$+ \sum_{a<b<c}(-1)^{a+b+c}\{-y_c.\psi([a,b]\hat{a}\hat{b}\hat{c}, t) \tag{5.21a}$$

$$+ y_b.\psi([a,c]\hat{a}\hat{b}\hat{c}, t) \tag{5.21b}$$

$$- y_a.\psi([b,c]\hat{a}\hat{b}\hat{c}, t)\} \tag{5.21c}$$

$$+ \sum_{a<b}(-1)^{a+b}\{y_b.(y_a.\psi(\hat{a}\hat{b}, t)) - y_a.(y_b.\psi(\hat{a}\hat{b}, t))\}. \tag{5.22}$$

The cancellation scheme is:

1) (5.17) and (5.20);

2) (5.18) and (5.22), since the action χ of $\mathcal{R}$ on $\mathcal{M}$ is a representation:

$$\chi([a,b]) = [\chi(a), \chi(b)], \qquad \forall\, a, b \in \mathcal{R}; \tag{5.23}$$

3) (5.19) and (5.21).

(Notice that $(\delta^{\mathrm{new}})^2 \neq 0$.) ■

Remark 5.24. The coboundary operator δ (5.10) does *not* reduce to the one of the Hochschild complex [2] when $\mathcal{R}$ is an associative algebra, even though the cochain spaces are identical in both cases.

Remark 5.25. When $\mathcal{M} = \mathcal{R}$ and the natural definition of representation is used, one arrives at a new complex by considering deformations of the quasiassociative algebra $\mathcal{R}$, exactly like the Hochschild complex on $C^\bullet(\mathcal{R}, \mathcal{R})$ is arrived at in the associative case [1]. This new complex is closely related to the Hochschild one, and it is still different from the one constructed above.

6 Dual Point of View, Homology

The extended complex (5.10) of the preceding Section was based on the notion of representation of a quasiassociative algebra $\mathcal{R}$ as a linear map $\chi : \mathcal{R} \to \mathrm{End}(\mathcal{M})$ satisfying the condition

$$\chi(a * b) = \chi(a)\chi(b). \tag{6.1}$$

There was a second version of representation, formula (5.7):

$$\chi(a * b) = -\chi(b)\chi(a); \tag{6.2}$$

this choice was left unexamined. Let's examine it now.

These two choices lead to two different formulae for the coboundary operator $\delta : C^1 \to C^2$,

$$\delta\psi(a,b) = \psi(a * b) - \chi(a)\psi(b), \tag{6.3}$$

$$\delta\psi(a,b) = \psi(a * b) + \chi(b)\psi(a). \tag{6.4}$$

The first direction was pursued in the preceding Section. The second one, as is easy to discover by considering the hypothetical map $\delta : C^2 \to C^3$, leads nowhere. Why is it so?

Let $\mathcal{M}^* = \mathrm{Hom}_K(\mathcal{M}, K)$ be the dual space to $\mathcal{M}$. Since $\mathcal{R}$ acts on $\mathcal{M}$, it also acts on $\mathcal{M}^*$ in the dual way:

$$\langle \chi^d(a)(m^*), m\rangle = -\langle m^*, \chi(a)(m)\rangle. \tag{6.5}$$

Hence,

$$\begin{aligned}\langle \chi^d(a * b)(m^*), m\rangle &= -\langle m^*, \chi(a * b)(m)\rangle \\ &= -\langle m^*, \chi(a)\chi(b)(m)\rangle = -\langle \chi^d(b)\chi^d(a)(m^*), m\rangle,\end{aligned}$$

so that

$$\chi^d(a * b) = -\chi^d(b)\chi^d(a). \tag{6.6}$$

Thus, our second version of representation, (6.2), is in fact dual to the first one, (6.1). Therefore, this definition is suited not for cohomology but for the dual object, homology. Defining the n-chains as

$$C_0 = C_0(\mathcal{R}, \mathcal{N}) = \mathcal{N}; \qquad C_n = C_n(\mathcal{R}, \mathcal{N}) = \mathcal{N} \otimes \mathcal{R}^{\otimes n}, \qquad n \in \mathbf{N}, \tag{6.7}$$

where $\mathcal{N}$ is a $\mathcal{R}$-module on which $\mathcal{R}$ acts according to formula (6.2):

$$\bar{n}^\bullet(a * b) = -(\bar{n}^\bullet a)^\bullet b, \qquad \bar{n} \in \mathcal{N}, \quad a, b \in \mathcal{R}, \tag{6.8}$$

in the suggestive notation of the right action, we define the differential $\partial : C_n \to C_{n-1}$ by the rule:

$$\begin{aligned} \partial(\bar{n} \otimes a_1 \otimes \ldots \otimes a_n) &= \sum_{i=1}^{n-1} (-1)^{i+1} \bar{n} \otimes \ldots \hat{i} \ldots a_i * a_n \\ &+ \sum_{1 \le i < j < n} (-1)^{i+j+1} \bar{n} \otimes [a_i, a_j] \ldots \hat{i} \ldots \hat{j} \ldots + \sum_{i=1}^{n-1} (-1)^{i+1} (n^\bullet a_i) \otimes \ldots \hat{i} \ldots, \\ \partial(C_0) = 0, \qquad \partial(\bar{n} \otimes a) &= \bar{n}^\bullet a. \end{aligned} \tag{6.9}$$

Since this formula satisfies the duality relation

$$\langle \partial \Psi, \psi \rangle = \langle \Psi, \delta \psi \rangle \tag{6.10}$$

for the case $\Psi \in C_n(\mathcal{R}, \mathcal{N}) \approx (C^n(\mathcal{R}, \mathcal{N}^*))^*$, $\psi \in C^{n-1}(\mathcal{R}, \mathcal{N}^*)$, we have $\partial^2 = 0$ as a matter of course; it is assumed that the chains considered are κ-skewsymmetric for some $\kappa \geq 2$, exactly like the cochains.

Remark 6.12. The Hochschild coboundary operator on $C^1 = \mathrm{Hom}(\mathcal{R}, \mathcal{M})$ acts by the rule:

$$\delta\psi(a_1, a_2) = a_1.\psi(a_2) - \psi(a_1 a_2) + \psi(a_1)^\bullet a_2, \tag{6.13}$$

where $\mathcal{R}$ is associative, $\mathcal{M}$ is an $\mathcal{R}$-bimodule, and the right action of $\mathcal{R}$ on $\mathcal{M}$ is an anti-action from the pont of view of our definition (6.2). We see that formulae (6.3) and (6.4) each contribute about half to the Hochschild formula (6.13). There must be some underlying reason for such split.

7 Differential Algebra Viewpoint

Suppose our basic ring K is a differential ring, with a derivation $\partial : K \to K$. Then the formula

$$[X, Y] = XY' - X'Y, \qquad (\cdot)' = \partial(\cdot), \quad X, Y \in K, \tag{7.1}$$

makes K into a Lie algebra $\mathcal{D}_1 = \mathcal{D}_1(K)$, the Lie algebra of vector fields. The bilinear form ω on $K \times K$,

$$\omega(X, Y) = XY''' \tag{7.2}$$

is skewsymmetric:

$$\omega(X, Y) \sim -\omega(Y, X), \tag{7.3}$$

and is a generalized 2-cocycle on $\mathcal{D}_1$:

$$\omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) \sim 0, \tag{7.4}$$

where $(\cdot) \sim 0$ means that $(\cdot) \in \operatorname{Im} \partial$.

When

$$K = k\left[x, x^{-1}\right] \tag{7.5}$$

and

$$\partial = d/dx, \tag{7.6}$$

k being some number field or such, the Lie algebra $\mathcal{D}_1$ is isomorphic to the centerless Virasoro algebra under identification

$$e_n = x^{1-n}\frac{d}{dx}, \qquad X = \sum_n X_n e_n. \tag{7.7}$$

As far as the Virasoro 2-cocycle is concerned, let

$$\mathrm{Res}: \; k\left[x, x^{-1}\right] \to k \tag{7.8}$$

be the map isolating the x^{-1}-coefficient, so that

$$\mathrm{Res} \circ \partial = 0.$$

Then

$$\begin{aligned} \mathrm{Res}(\omega(e_n, e_m)) &= Res\left(x^{1-n}(1-m)(-m)(-1-m)x^{-2-m}\right) \\ &= \delta^0_{n+m}(n+1)n(n-1) = \left(n^3 - n\right)\delta^0_{n+m}. \end{aligned} \tag{7.9}$$

Below we construct a quasiassociative structure on $K = k\left[x, x^{-1}\right]$ and the corresponding generalized 2-cocycle on it, so that formulae (1.5) and (3.11) are recovered as localizations.

Let

$$\mathcal{O} = x\frac{d}{dx} - 1 \tag{7.10}$$

and set

$$u * v = (1 - \epsilon\mathcal{O})^{-1} x^{-1} u (1 - \epsilon\mathcal{O})\mathcal{O}(v), \tag{7.11}$$

$$\hat{\Omega}(u, v) = x^{-3}\mathcal{O}^2(1 + \epsilon\mathcal{O})(u) \cdot v. \tag{7.12}$$

Since

$$\mathcal{O}\left(x^{1-q}\right) = -qx^{1-q}, \tag{7.13}$$

we get

$$\begin{aligned} x^{1-p} * x^{1-q} &= (1-\epsilon\mathcal{O})^{-1}\left(x^{-1}x^{1-p}(1+\epsilon q)(-q)x^{1-q}\right) \\ &= -q(1-\epsilon q)(1-\epsilon\mathcal{O})^{-1}x^{1-p-q} = -\frac{q(1+\epsilon q)}{1+\epsilon(p+q)}x^{1-p-q}. \end{aligned} \tag{7.14}$$

This is formula (1.5). It implies that we have a correct quasiassociative multiplication on $k\left[x, x^{-1}\right]$, with

$$u * v - v * u = uv' - u'v. \tag{7.15}$$

The 2-cocycle story is more interesting. Recall how the notion of the *generalized* 2-cocycle on a Lie algebra, equation (7.4), appears: from the classification of *affine* Hamiltonian operators, with the linear part being attached to a Lie algebra, say $\mathcal{G}$, and the constant part being a *generalized* 2-cocycle on this Lie algebra [4]. Aposteriori one can put all this into a variational complex ([4], p. 204) $\delta :\ \mathrm{Diff}\,(\wedge^n\mathcal{G}, K)_v \to\ \mathrm{Diff}\,(\wedge^{n+1}\mathcal{G}, K)_v$, where subscript "$v$" signifies that differential forms differing by $Im\ \partial$ are to be identified; the generalized 2-cocycle condition (7.4) is then simply

$$\delta\omega(X, Y, Z) \sim 0. \tag{7.16}$$

We shall now apply the same variational leap-forward to the complex $C^\bullet(\mathcal{R}, K)$ of Section 5, considering cochains modulo Im ∂. A generalized 2-cocycle $\hat{\Omega}$ then satisfies the differential version of the equality (3.4):

$$\delta\hat{\Omega}(u, v, w) = \hat{\Omega}(v, u * w) - \hat{\Omega}(u, v * w) + \hat{\Omega}([u, v], w) \sim 0. \tag{7.17}$$

It is unclear to me at the moment exactly what question such a variational 2-cocycle answers to, and repeated appeals to noncommutative differential geometry in the sense of Allan Connes haven't helped so far; nevertheless, we have

Proposition 7.19. *(i) Let $\hat{\Omega}$ be a generalized 2-cocycle on a differential quasiassociative algebra $\mathcal{R}$. Then*

$$\hat{\omega}(u, v) = \hat{\Omega}(u, v) - \hat{\Omega}(v, u) \tag{7.20}$$

is a generalized 2-cocycle on the Lie algebra $Lie(\mathcal{R})$;
(ii) The symplectic form on $T^*\mathcal{R}$ is a generalized 2-cocycle on $T^*\mathcal{R}$.

Proof. (i) We have,

$$\begin{aligned} \hat{\omega}([u, v], w) &= \hat{\Omega}([u, v], w) - \hat{\Omega}(w, [u, v]) \\ &\overset{[\text{by } (7.18)]}{\sim}\ \hat{\Omega}(u, vw) - \hat{\Omega}(v, uw) - \hat{\Omega}(w, uv - vu). \end{aligned} \tag{7.21}$$

Hence,

$$\begin{aligned} \hat{\omega}([u, v], w) + c.p. &\sim (\hat{\Omega}(u, vw) + c.p.) - (\hat{\Omega}(v, uw) + c.p.) - (\hat{\Omega}(w, uv - vu) + c.p.) \\ &= (\hat{\Omega}(w, uv) + c.p.) - (\hat{\Omega}(w, vu) + c.p) - (\hat{\Omega}(w, uv - vu) + c.p.) = 0; \end{aligned}$$

(ii) By formula (1.1), $T^*\mathcal{R}$ has the multiplication

$$\begin{pmatrix} u \\ \bar{u} \end{pmatrix} * \begin{pmatrix} v \\ \bar{v} \end{pmatrix} = \begin{pmatrix} u * v \\ u * \bar{v} \end{pmatrix}, \qquad u, v \in \mathcal{R}, \quad \bar{u}, \bar{v} \in \mathcal{R}^*, \tag{7.22a}$$

where

$$\langle u * \bar{v}, w \rangle \sim -\langle \bar{v}, u * w \rangle. \tag{7.22b}$$

The symplectic form $\hat{\Omega}$ is

$$\hat{\Omega}\left(\begin{pmatrix} u \\ \bar{u} \end{pmatrix}, \begin{pmatrix} v \\ \bar{v} \end{pmatrix} \right) = \langle \bar{u}, v \rangle - \langle \bar{v}, u \rangle.$$

Hence,

$$\begin{aligned} \hat{\Omega}\left(\begin{pmatrix} v \\ \bar{v} \end{pmatrix}, \begin{pmatrix} u \\ \bar{u} \end{pmatrix} * \begin{pmatrix} w \\ \bar{w} \end{pmatrix} \right) &= \langle \bar{v}, u * w \rangle - \langle u * \bar{w}, v \rangle \\ &\sim \langle \bar{v}, u * w \rangle + \langle \bar{w}, u * v \rangle, \end{aligned} \tag{7.23a}$$

$$-\hat{\Omega}\left(\begin{pmatrix} u \\ \bar{u} \end{pmatrix}, \begin{pmatrix} v \\ \bar{v} \end{pmatrix} * \begin{pmatrix} w \\ \bar{w} \end{pmatrix} \right) \sim -\langle \bar{u}, v * w \rangle - \langle \bar{w}, v * u \rangle, \tag{7.23b}$$

$$\begin{aligned} \hat{\Omega}\left(\left[\begin{pmatrix} u \\ \bar{u} \end{pmatrix}, \begin{pmatrix} v \\ \bar{v} \end{pmatrix} \right], \begin{pmatrix} w \\ \bar{w} \end{pmatrix} \right) &= \Omega\left(\begin{pmatrix} [u, v] \\ u * \bar{v} - v * \bar{u} \end{pmatrix}, \begin{pmatrix} w \\ \bar{w} \end{pmatrix} \right) \\ &= \langle u * \bar{v} - v * \bar{u}, w \rangle - \langle \bar{w}, [u, v] \rangle \\ &\sim \ -\langle \bar{v}, u * w \rangle + \langle \bar{u}, v * w \rangle - \langle \bar{w}, u * v - v * u \rangle. \end{aligned} \tag{7.23c}$$

Adding the expressions (7.23a-c) up, we get zero. ■

Let us now verify that $\hat{\Omega}$ given by formula (7.13) is indeed a generalized 2-cocycle. We have:

$$\begin{aligned} 1) \quad \hat{\Omega}(v, uw) &= x^{-3}\mathcal{O}^2(1 + \epsilon\mathcal{O})(v) \cdot (1 - \epsilon\mathcal{O})^{-1} x^{-1} u (1 - \epsilon\mathcal{O})\mathcal{O}(w) \\ &\sim [1 + \epsilon(\mathcal{O} + 3)]^{-1} x^{-3}\mathcal{O}^2(1 + \epsilon\mathcal{O})(v) \cdot x^{-1} u (1 - \epsilon\mathcal{O})\mathcal{O}(w) \\ &\sim \left\{ -(\mathcal{O} + 3)[1 + \epsilon(\mathcal{O} + 3)] x^{-1} u [1 + \epsilon(\mathcal{O} + 3)]^{-1} x^{-3}\mathcal{O}^2(1 + \epsilon\mathcal{O})(v) \right\} \cdot w, \end{aligned} \tag{7.24}$$

where we used the universal relation

$$(1)A(2) \sim A^\dagger(1) \cdot (2) \tag{7.25}$$

for the adjoint operator, and the particular relation

$$\mathcal{O}^\dagger = -(\mathcal{O} + 3) \tag{7.26}$$

for our operator $\mathcal{O} = x\dfrac{d}{dx} - 1$.

Now, since

$$(\mathcal{O} + 3)x^{-3} = x^{-3}\mathcal{O}, \tag{7.27}$$

formula (7.24) can be rewritten as

$$\left\{-x^{-4}(\mathcal{O}-1)[1+\epsilon(\mathcal{O}-1)]u\mathcal{O}^2(v)\right\}\cdot w; \tag{7.28a}$$

$$2)\quad -\hat{\Omega}(u,vw)\sim\left\{x^{-4}(\mathcal{O}-1)[1+\epsilon(\mathcal{O}-1)]v\mathcal{O}^2(u)\right\}\cdot w; \tag{7.28b}$$

$$3)\quad \hat{\Omega}([u,v],w)=\left\{x^{-3}\mathcal{O}^2(1+\epsilon\mathcal{O})(uv'-u'v)\right\}\cdot w. \tag{7.28c}$$

Adding up the expressions (7.28a-c), we arrive at the equivalent relation to be verified:

$$x^{-3}\mathcal{O}^2(1+\epsilon\mathcal{O})(uv'-u'v)=x^{-4}(\mathcal{O}-1)[1+\epsilon(\mathcal{O}-1)]\left[v\mathcal{O}^2(u)-u\mathcal{O}^2(v)\right]. \tag{7.29}$$

Since

$$x^{-1}(\mathcal{O}-1)=\mathcal{O}x^{-1}, \tag{7.30}$$

equality (7.29) reduces to

$$\mathcal{O}(uv'-u'v)=x^{-1}\left[u\mathcal{O}^2(v)-v\mathcal{O}^2(u)\right]. \tag{7.31}$$

Now,

$$\mathcal{O}^2=x^2\frac{d^2}{dx^2}-x\frac{d}{dx}+1, \tag{7.32}$$

so that

$$\begin{aligned}x^{-1}\left[u\mathcal{O}^2(v)-v\mathcal{O}^2(w)\right]&=u(xv''-v')-v(xu''-u')\\ &=x(uv''-u''v)-(uv'-u'v)=\left(x\frac{d}{dx}-1\right)(uv'-u'v)=\mathcal{O}(uv'-u'v).\end{aligned}$$

It remains to perform the last step: to calculate $\mathrm{Res}\,\hat{\Omega}\left(x^{1-p},x^{1-q}\right)$ and to compare the result with the formulae (3.11,12). We have:

$$\hat{\Omega}\left(x^{1-p},x^{1-q}\right)=x^{-3}\mathcal{O}^2(1+\epsilon\mathcal{O})\left(x^{1-p}\right)\cdot x^{1-q}\overset{\text{[by (7.14)]}}{=}(-p)^2(1-\epsilon p)x^{-1-p-q}, \tag{7.33}$$

so that

$$\mathrm{Res}\,\hat{\Omega}\left(x^{1-p},x^{1-q}\right)=p^2(1-\epsilon p)\delta^0_{p+q}=-\epsilon\left(p^3-\epsilon^{-1}p^2\right)\delta^0_{p+q}. \tag{7.34}$$

We see that we have to multiply $\hat{\Omega}$ by $-\frac{1}{2}\epsilon^{-1}$, and also to add to it the trivial 2-cocycle proportional to the $*$ product. From formula (7.15) we find:

$$\mathrm{Res}\,x^{-2}\left(x^{1-p}*x^{1-q}\right)=p(1-\epsilon p)\delta^0_{p+q}. \tag{7.35}$$

Thus, the correctly normalized generalized 2-cocycle has the form

$$\hat{\Omega}^{\mathrm{new}}(u,v)=-\frac{1}{2}\epsilon^{-1}x^{-3}\mathcal{O}^2(1+\epsilon\mathcal{O})(u)\cdot v-\frac{1}{2}x^{-2}(1-\epsilon\mathcal{O})^{-1}x^{-1}u(1-\epsilon\mathcal{O})\mathcal{O}(v). \tag{7.36}$$

Remark 7.36. Consider the Lie algebra $\mathcal{D}_n=\mathcal{D}_n(K)$ "of vector fields on $\mathbf{R}^n$", with the commutator

$$[X,Y]^i=\sum_{s=1}^{n}(X^sY^i{}_{,s}-Y^sX^i{}_{,s}),\qquad X,Y\in K^n, \tag{7.37}$$

where

$$(\cdot)_{,s} = \partial_s(\cdot), \tag{7.38}$$

and $\partial_1, \dots, \partial_n : K \to K$ are n commuting derivations. Localizing K as $k[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$ and taking as the basis of $\mathcal{D}_n(K)$

$$e^i_\sigma = x^{1_i - \sigma}\partial_i = x_1^{-\sigma_1} \dots x_n^{-\sigma_n} x_i \partial_i, \qquad \sigma \in \mathbf{Z}^n, \quad \partial_i = \partial/\partial x_i, \tag{7.39}$$

we find the n-dimensional analog of the centerless Virasoro algebra:

$$[e^i_\sigma, e^j_\nu] = (\delta_{ij} - \nu_i)e^j_{\sigma+\nu} - (\delta_{ij} - \sigma_j)e^i_{\sigma+\nu}. \tag{7.40}$$

This Lie algebra does not seem to have a quasiassoactive representation of the form (1.5) for $n > 1$, but it does have a quasiassociative representation generalizing formula (2.11):

$$e^i_\sigma * e^j_\nu = (\lambda\delta_{ij} - \nu_i)e^i_{\sigma+\nu}, \qquad \lambda = \text{const}. \tag{7.41}$$

Appendix 1. Virasoro Algebra Does Not Come from an Associative One

Suppose we have a $\mathbf{Z}$-graded multiplication on the basis $\{e_p \mid p \in G$, a commutative ring$\}$, of the form

$$e_i e_j = g(i, j)e_{i+j}, \tag{A1.1}$$

such that

$$e_i e_j - e_j e_i = (i - j)e_{i+j}, \qquad \forall\, i, j \in G, \tag{A1.2}$$

and

$$(e_i e_j)e_k = e_i(e_j e_\kappa), \qquad \forall\, i, j, \kappa \in G. \tag{A1.3}$$

Let us show that such representation is impossible.

We first rewrite the boundary condition (A1.2) as

$$g(i, j) - g(j, i) = i - j. \tag{A1.4}$$

Next, rewrite the associativity condition (A1.3) as

$$g(i, j)g(i + j, \kappa) = g(j, \kappa)g(i, j + \kappa). \tag{A1.5}$$

Now, set $j = \kappa = 0$ in formula (A1.5):

$$g(i, 0)[g(i, 0) - g(0, 0)] = 0. \tag{A1.6}$$

Further, set $j = i = 0$ in formula (A1.5):

$$g(0, \kappa)[g(0, \kappa) - g(0, 0)] = 0. \tag{A1.7}$$

Assume that G has no zero divisors. From formula (A1.6) we find:

$$g(i, 0) = 0 \quad \text{or} \quad g(0, 0), \tag{A1.8}$$

while formula (A1.7) yields:

$$g(0, \kappa) = 0 \quad \text{or} \quad g(0, 0). \tag{A1.9}$$

The last two equations contradict the boundary condition (A1.4):

$$g(r, 0) - g(0, r) = r.$$

Appendix 2. Semidirect Sums of Quasiassociative Algebras

Let $\mathcal{R}$ and $\mathcal{U}$ be quasiassociative algebras, $\mathcal{G} = Lie(\mathcal{R})$, $\mathcal{H} = Lie(\mathcal{U})$. Let $\chi : \mathcal{G} \to \mathrm{Der}(\mathcal{H})$ be a representation of $\mathcal{G}$. The semidirect sum Lie algebra $\mathcal{G} \underset{\chi}{\bowtie} \mathcal{H}$ is the vector space $\mathcal{G} \oplus \mathcal{H}$ with the commutator

$$\left[\begin{pmatrix} a \\ u \end{pmatrix}, \begin{pmatrix} b \\ v \end{pmatrix}\right] = \begin{pmatrix} [a,b] \\ a.v - b.u + [u,v] \end{pmatrix}, \qquad a, b \in \mathcal{G}, \quad u, v \in \mathcal{H}. \tag{A2.1}$$

Does the Lie algebra $\mathcal{G} \underset{\chi}{\bowtie} \mathcal{H}$ have a quasiassociative representation?

Proposition A2.2. *Let* $\chi :\ Lie(\mathcal{R}) \to \mathrm{Der}(\mathcal{U})$ *be a representation. Define the semidirect sum* $\mathcal{R} \underset{\chi}{\bowtie} \mathcal{U}$ *as the space* $\mathcal{R} \oplus \mathcal{U}$ *with the multiplication*

$$\begin{pmatrix} a \\ u \end{pmatrix} * \begin{pmatrix} b \\ b \end{pmatrix} = \begin{pmatrix} a * b \\ a.v + u * v \end{pmatrix}, \qquad a, b \in \mathcal{R}, \quad u, v \in \mathcal{U}. \tag{A2.3}$$

Then this multiplication is quasiassociative.

Proof. Dropping the $*$ notation for brevity, we have

$$\begin{pmatrix} a \\ u \end{pmatrix}\left(\begin{pmatrix} b \\ v \end{pmatrix}\begin{pmatrix} c \\ w \end{pmatrix}\right) = \begin{pmatrix} a \\ u \end{pmatrix}\begin{pmatrix} bc \\ b.w + vw \end{pmatrix} = \begin{pmatrix} a(bc) \\ a.(b.w + vw) + u(b.w + vw) \end{pmatrix},$$

$$\left(\begin{pmatrix} a \\ u \end{pmatrix}\begin{pmatrix} b \\ v \end{pmatrix}\right)\begin{pmatrix} c \\ w \end{pmatrix} = \begin{pmatrix} ab \\ a.v + uv \end{pmatrix}\begin{pmatrix} c \\ w \end{pmatrix} = \begin{pmatrix} (ab)c \\ (ab).w + (a.v + uv)w \end{pmatrix}.$$

Thus, we need to verify that

$$\begin{aligned} & a.(b.w + vw) + u(b.w + vw) - (ab).w - (a.v + uv)w \\ & \quad = b.(a.w + uw) + v(a.w + uw) - (ba).w - (b.u + vu)w. \end{aligned}$$

This can be rewritten as $0 \overset{?}{=}$

$$a.(b.w) - b.(a.w) - ((ab).w - (ba).w) \tag{A2.4a}$$

$$+\, a.(vw) - (a.v)w - v(a.w) \tag{A2.4b}$$

$$+\, u(b.w) + (b.u)w - b.(uw) \tag{A2.4c}$$

$$+\, u(vw) - (uv)w - v(uw) + (vu)w. \tag{A2.4d}$$

The first sum vanishes since χ is a representation of $Lie\ (\mathcal{R})$; the second and third sums vanish since $Im(\chi) \subset \mathrm{Der}(\mathcal{U})$; the fourth sum vanishes since $\mathcal{U}$ is quasiassociative. ■

Corollary A2.5. *If* $\mathcal{U}$ *is abelian and* $\chi :\ Lie(\mathcal{R}) \to \mathrm{End}(\mathcal{U})$ *is a representation, then* $\mathcal{R} \underset{\chi}{\bowtie} \mathcal{U}$ *is quasiassociative.*

Proof. $\mathrm{Der}(\mathcal{U}) = \mathrm{End}(\mathcal{U})$ for an abelian $\mathcal{U}$. ■

Example A2.6. Consider the Ehrenfest Lie algebra $\mathcal{G}(A)$, where A is an arbitrary matrix, and the commutators between basis elements are ([6], p. 274):

$$[e_i, e_j] = [\bar{e}_i, \bar{e}_j] = 0, \qquad [e_i, \bar{e}_j] = A_{ji}\bar{e}_j. \tag{A2.7}$$

In this case both $\mathcal{R}$ and $\mathcal{U}$ are vector spaces with trivial multiplication,

$$a * b = 0, \qquad u * v = 0, \qquad \forall\, a, b \in \mathcal{R}, \quad u, v \in \mathcal{U}, \tag{A2.8}$$

and the representation χ acts by the fule

$$e_i.\bar{e}_j = A_{ji}\bar{e}_j, \tag{A2.9}$$

It *is* a representation of the abelian Lie algebra $Lie(\mathcal{R})$, since

$$e_i.(e_j.\bar{e}_\kappa) = e_j.(e_i.\bar{e}_\kappa) = A_{\kappa i}A_{\kappa j}\bar{e}_\kappa. \tag{A2.10}$$

Hence, the Ehrenfest Lie algebra $\mathcal{G}(A)$ (A2.7) comes out of the following quasiassociative multiplication:

$$e_ie_j = \bar{e}_i\bar{e}_j = \bar{e}_ie_j = 0, \qquad e_i\bar{e}_j = A_{ji}\bar{e}_j. \tag{A2.11}$$

Remark A2.12. Proposition A2.2 shows that

$$Lie(R)\underset{\chi}{\rhd\!\!\!<} Lie(\mathcal{U}) = Lie(\mathcal{R}\underset{\chi}{\rhd\!\!\!<}\mathcal{U}) \tag{A2.13}$$

when $\mathcal{U}$ is abelian. Otherwise formula (A2.13) is not necessarily true since $\mathrm{Der}(\mathcal{U})$ is, in general, smaller than $\mathrm{Der}(Lie(\mathcal{U}))$:

Proposition A2.14. *(i)* $\mathrm{Der}(\mathcal{U}) \subset \mathrm{Der}(Lie(\mathcal{U}))$*;*
(ii) If $\mathrm{Int}(\mathcal{U}) \subset \mathrm{Der}(\mathcal{U})$ *then* $\mathcal{U}$ *is associative. (Here* $\mathrm{Int}(\mathcal{U})$ *denotes the space of maps* $\{\mathrm{ad}_u :\ \mathcal{U} \to \mathcal{U}\ |u \in \mathcal{U}\}$*.)*

Proof. (i) is well-known to be true for any algebra, not necessarily associative or quasiassociative one;

(ii) ad_u is a derivation of $Lie(\mathcal{U})$ no matter whether $\mathcal{U}$ is quasiassociative or not. For ad_u to be a derivation of $\mathcal{U}$, we must have, for any $u, v, w \in \mathcal{U}$:

$$0 = ad_u(vw) - (ad_u(v))w - vad_u(w) = u(vw) - (vw)u - (uv - vu)w - v(uw - wu)$$

$$= u(vw) - (uv)w - v(uw) + (vu)w \tag{A2.15a}$$

$$- (vw)u + v(wu). \tag{A2.15b}$$

The first sum vanishes since $\mathcal{U}$ is quasiassociative. The second sum vanishes iff $\mathcal{U}$ is associative. ∎

Appendix 3. Lie Algebras of Vector Fields on Lie Groups

Formula

$$X * Y = XY', \qquad X, Y \in C^{\infty}(S^{-1}), \quad ' = \frac{d}{dz}, \tag{A3.1}$$

provides a quasiassociative structure on the Lie algebra of vector fields on the circle, $\mathcal{D}(S^1)$. Formula [5]

$$(X * Y)^i = \sum_s X^s Y^i{}_{,s} \tag{A3.2}$$

provides a quasiassociative structure on the Lie algebra of vector fields on $\mathbf{R}^n, \mathcal{D}(\mathbf{R}^n)$. This suggests that for some manifolds, similar structure exists for their Lie algebras of vector fields. (This will be proven below for $GL(n, \mathbf{R})$ and $GL(n, \mathbf{C})$.) The parallelizable manifolds are the simplest, and Lie groups are simpler still.

Proposition A3.3. *Let $\mathcal{R}$ be a finite-dimensional quasiassociative algebra over $\mathbf{R}$, $\mathcal{G} = Lie(\mathcal{R})$, and G a connected Lie group with the Lie algebra $\mathcal{G}$. Then the Lie algebra of vector fields on G, $\mathcal{D}(G)$, has a quasiassociative representation.*

Proof. Let (e_i) be a basis in $\mathcal{R}$. Then

$$e_i e_j = \sum_s \theta^s_{ij} e_s, \tag{A3.4}$$

with some structure constants $\theta^s_{ij} \in \mathbf{R}$. The quasiassociativity condition

$$(e_i e_j)e_\kappa - e_i(e_j e_\kappa) = (e_j e_i)e_\kappa - e_j(e_i e_\kappa), \qquad \forall\, i, j, \kappa, \tag{A3.5}$$

translates into the equality

$$\sum_s \left(\theta^s_{jk}\theta^r_{is} - \theta^s_{ij}\theta^r_{sk}\right) = \sum_s \left(\theta^s_{ik}\theta^r_{js} - \theta^s_{ji}\theta^r_{sk}\right), \tag{A3.6}$$

or

$$\sum_s \left(\theta^s_{jk}\theta^r_{is} - \theta^s_{ik}\theta^r_{js}\right) = \sum_s c^s_{ij}\theta^r_{sk}, \tag{A3.7}$$

where

$$c^s_{ij} = \theta^s_{ij} - \theta^s_{ji} \tag{A3.8}$$

are the structure constants of the Lie algebra $\mathcal{G} = Lie(\mathcal{R})$:

$$[e_i, e_j] = e_i e_j - e_j e_i = \sum_s c^s_{ij} e_s. \tag{A3.9}$$

Denote by $\hat{e}_i$ the left-invariant vector fields on G generated by the elements $e_i \in \mathcal{G}$, so that

$$\hat{e}_i\hat{e}_j - \hat{e}_j\hat{e}_i = \sum_j c^s_{ij}\hat{e}_s. \tag{A3.10}$$

In this basis, every vector field on G can be identified with a vector from $C^\infty(G)^{\dim(G)}$:

$$X \in \mathcal{D}(G) \;\Rightarrow\; X = \sum_i X^i \hat{e}_i, \qquad X^i \in C^\infty(G). \tag{A3.11}$$

For $X = \sum X^i \hat{e}_i$, $Y = \sum Y^j \hat{e}_j \in \mathcal{D}(G)$, set

$$(X * Y)^r = \sum_\alpha X^\alpha \hat{e}_\alpha(Y^r) + \sum_{\alpha\beta} X^\alpha Y^\beta \theta^r_{\alpha\beta}. \tag{A3.12}$$

We are going to show that this multiplication makes $\mathcal{D}(G) \approx C^\infty(G)^{\dim(G)}$ into a quasiassociative algebra; the boundary conditions are satisfied since

$$\begin{aligned}\sum_r (X * Y - Y * X)^r \hat{e}_r &= \sum_{\alpha r} \left[X^\alpha \hat{e}_\alpha(Y^r) - Y^\alpha \hat{e}_\alpha(X^r)\right] \hat{e}_r \\ &+ \sum_{\alpha\beta r} X^\alpha Y^\beta c^r_{\alpha\beta} \hat{e}_r = \sum_{\alpha\beta} \left[X^\alpha \hat{e}_\alpha, Y^\beta \hat{e}_\beta\right] = [X, Y].\end{aligned} \tag{A3.13}$$

Now,

$$\begin{aligned}(X(YZ))^r &= \sum_\alpha X^\alpha \hat{e}_\alpha((YZ)^r) + \sum_{\alpha\beta} \theta^r_{\alpha\beta} X^\alpha (YZ)^\beta \\ &= \sum_\alpha X^\alpha \hat{e}_\alpha \left(\sum_\mu Y^\mu \hat{e}_\mu(Z^r) + \sum_{\mu\nu} \theta^r_{\mu\nu} Y^\mu Z^\nu\right) \\ &+ \sum_{\alpha\beta} \theta^r_{\alpha\beta} X^\alpha \left(\sum_\mu Y^\mu \hat{e}_\mu(Z^\beta) + \sum_{\mu\nu} \theta^\beta_{\mu\nu} Y^\mu Z^\nu\right) \\ &= \sum_{\alpha\mu} X^\alpha \hat{e}_\alpha(Y^\mu) \hat{e}_\mu(Z^r) + \sum_{\alpha\mu} X^\alpha Y^\mu \hat{e}_\alpha \hat{e}_\mu(Z^r) \\ &+ \sum_{\alpha\mu\nu} X^\alpha \theta^r_{\mu\nu} \left(\hat{e}_\alpha(Y^\mu) Z^\nu + Y^\mu \hat{e}_\alpha(Z^\nu)\right) \\ &+ \sum_{\alpha\mu\nu} \theta^r_{\alpha\nu} X^\alpha Y^\mu \hat{e}_\mu(Z^\nu) + \sum_{\alpha s\mu\nu} \theta^r_{\alpha s} \theta^s_{\mu\nu} X^\alpha Y^\mu Z^\nu,\end{aligned} \tag{A3.14}$$

$$\begin{aligned}((XY)Z)^r &= \sum_\mu (XY)^\mu \hat{e}_\mu(Z^r) + \sum_{s\nu} \theta^r_{s\nu} (XY)^s Z^\nu = \sum_{\mu\alpha} X^\alpha \hat{e}_\alpha(Y^\mu) \hat{e}_\mu(Z^r) \\ &+ \sum_{\mu\alpha\beta} \theta^\mu_{\alpha\beta} X^\alpha Y^\beta \hat{e}_\mu(Z^r) + \sum_{s\nu} \theta^r_{s\nu} \left(\sum_\alpha X^\alpha \hat{e}_\alpha(Y^s) + \sum_{\alpha\beta} \theta^s_{\alpha\beta} X^\alpha Y^\beta\right) Z^\nu.\end{aligned} \tag{A3.15}$$

Thus,

$$\begin{aligned}&(X(YZ) - (XY)Z)^r \\ &\quad = \sum_{\alpha\mu} X^\alpha Y^\mu \hat{e}_\alpha \hat{e}_\mu(Z^r)\end{aligned} \tag{A3.16a}$$

$$+\sum_{\alpha\mu\nu} X^\alpha Y^\mu \theta^r_{\mu\nu}\hat{e}_\alpha(Z^\nu) + \sum_{\alpha\mu\nu} X^\alpha Y^\mu \theta^r_{\alpha\nu}\hat{e}_\mu(Z^\nu) \tag{A3.16b}$$

$$+\sum_{\alpha s\mu\nu} X^\alpha Y^\mu Z^\nu \left(\theta^r_{\alpha s}\theta^s_{\mu\nu} - \theta^r_{s\nu}\theta^s_{\alpha\mu}\right) \tag{A3.16c}$$

$$-\sum_{\mu\alpha\beta} \theta^\mu_{\alpha\beta} X^\alpha Y^\beta \hat{e}_\mu(Z^r). \tag{A3.16d}$$

Interchanging X and Y, subtracting the resulting expressions, noticing that (A3.16b) is symmpletric in (X, Y), and using formulae (A3.10,8), we arrive at the following identity to be verified: $0 \overset{?}{=}$

$$= \sum_{\alpha\mu s} X^\alpha Y^\mu \left(\theta^s_{\alpha\mu} - \theta^s_{\mu\alpha}\right)\hat{e}_s(Z^r) \tag{A3.17a}$$

$$+\sum_{\alpha s\mu\nu} X^\alpha Y^\mu Z^\nu \left(\theta^r_{\alpha s}\theta^s_{\mu\nu} - \theta^v_{s\nu}\theta^s_{\alpha\mu} - \theta^r_{\mu s}\theta^s_{\alpha\nu} + \theta^r_{s\nu}\theta^s_{\mu\alpha}\right) \tag{A3.17b}$$

$$-\sum_{\mu\alpha\beta} \left(\theta^\mu_{\alpha\beta} - \theta^\mu_{\beta\alpha}\right) X^\alpha Y^\beta \hat{e}_\mu(Z^r). \tag{A3.17c}$$

The expressions (A3.17a) and (A3.17c) cancel each other out. The sum (A3.17b) vanishes due to the quasiassociativity condition (A3.6). ■

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